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Note

A New Technique for Solving Parker-type Wind Equations

I. Introduction

In recent years, the astrophysical community has come to learn of the existence of extremely energetic and dynamic phenomena on or near the surface of compact objects, such as neutron stars. Sources ranging in scale from the local (galactic) X-ray bursters [1] to the distant (cosmological) quasars [2] have shown evidence of violent, radiatively driven mass expulsion. Invariably, the duration of this mass loss has been long compared to the relevant dynamic time scale, and one is therefore led to consider the structure of quasi-static flows analogous to the solar wind studied extensively by Parker [3] in the 1960's.

It is well known that the differential equations describing time-independent mass outflux from a compact source possess a singularity (the "critical point") where the fluid attains a velocity equal to the local sound speed (see Section II). On rare occasions, the equations admit an analytic solution; as a rule, however, numerical techniques must be employed to integrate through this singularity, and traditionally that is where the difficulties have been encountered. The various authors who have had to deal with this problem, have usually employed a customized technique that at times has sacrificed accuracy for ease of use (see Section II for an example). Matching the solution branches above and below the critical point is crucial to the process of finding the surface boundary conditions (which can be directly compared to the observations) as a function of the physical conditions at the base of the flow. In Section III of this paper, a new technique will be presented which should make this matching accurate and easily attainable.

II. The Equations and Topology of the Solutions

The relevant fluid equations are the continuity equation (conservation of mass), the equations of momentum and energy conservation, and the equation of heat transport, in Eulerian form, under the constraint of time independence. We shall assume for the sake of clarity that the flow is spherically symmetric, although the procedure can quite simply be adapted to other geometries. The exact form of the equations depends on the physical conditions (such as the equation of state and photon mean free path), but the expressions can generally be reduced to a set of

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two simultaneous, first-order differential equations in two unknowns (together with the constitutive relations: the equation of state, the internal energies of the gas and radiation, and the radiative opacities). For illustrative purposes, we shall take the fluid velocity v and the gas temperature T to be the dependent variables and the radius r to be the independent variable. The set of coupled differential equations can then be written in the form

$$\frac{dv}{dr}=\frac{N(r, v, T, \dot{M}, M)}{D(v, T)}, \tag{1}$$

and

$$\frac{dT}{dr}=H(r, v, T, \dot{M}, M), \tag{2}$$

where $\dot{M}$ is the total mass flux carried by the wind, M is the mass of the compact object, and

$$D \equiv v^2 - c_s^2. \tag{3}$$

Here, $c_s(T)$ is either the adiabatic or isothermal sound speed, depending on the conditions. The critical point corresponds to the level in the flow where $D=0$ (i.e., where $v=c_s$, as indicated above). In order for the solution to be regular at this point, it is necessary that N also vanish there.

The topology of the solutions to Eqs. (1) and (2) is shown schematically in Fig. 1 (see, e.g., [4]). There are six families of solutions, two of which are transonic (i.e., $v=c_s$ at some critical radius r_c). In most applications, the flow velocity is much smaller than the particle thermal velocity deep in the envelope where the matter is approximately in hydrostatic equilibrium. The flow must eventually attain escape velocity, however, and it therefore has a velocity in excess of the sound speed at sufficiently large radii. As such, the appropriate solutions for a study of quasi-static mass loss are those forming the transonic branch labelled "1" in Fig. 1.

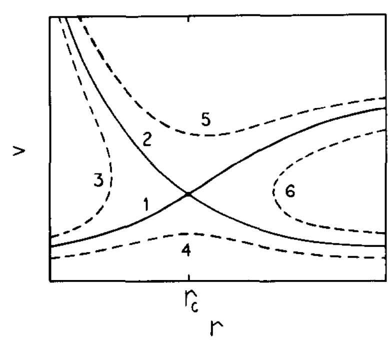

FIG. 1. The topology of solutions to (the wind) Eqs. (1) and (2). Of the six families labelled 1 to 6, only 1 and 2 are transonic.

To ensure that matching trajectories are joined at r_c, the normal procedure is to start the integration (inwards and outwards) at that point. However, Taylor series expansions for v and T are difficult to use for this purpose, as can be seen immediately by inspection of Eq. (1). Evidently, $(dv/dr)|_{r_c}$ cannot be determined until after the solution is known. One method that has been employed [5] amounts to putting

$$v = c_s(r_c) + \frac{N}{D}(r - r_c), \tag{4}$$

and

$$T = T(r_c) + H\Big|_{r_c}(r - r_c) + \frac{1}{2}\frac{dH}{dr}\Big|_{r_c}(r - r_c)^2, \tag{5}$$

but since both N and $D \to 0$ at r_c, the integration can become unstable, thus leading to a loss of accuracy across the singularity. This problem can be circumvented by the technique outlined in the following section.

III. Method of Solution

Let us define a function

$$\Phi(v, T) \equiv \frac{1}{2}\left(v + \frac{c_s^2}{v}\right). \tag{6}$$

Then, differentiating once with respect to r gives

$$\frac{d\Phi}{dr} = \frac{1}{2v^2}(v^2 - c_s^2)\frac{dv}{dr} + \frac{c_s}{v}\frac{dc_s}{dr}, \tag{7}$$

which can further be reduced to the form

$$\frac{d\Phi}{dr} = \frac{N}{2v^2} + \frac{c_s H}{v}\frac{dc_s}{dT}. \tag{8}$$

This equation has no singularity (except at the origin), so that substitution of Φ for v as one of the dependent variables yields a pair of coupled differential equations ((2) and (8)) whose solution throughout the envelope may be readily obtained by use of an implicit integration scheme.

As one might expect, the properties of Φ are those that define the two transonic solutions (see Fig. 1). Its first two partial derivatives with respect to v are

$$\frac{\partial\Phi}{\partial v} = \frac{1}{2}\left(1 - \frac{c_s^2}{v^2}\right), \tag{9}$$

and

$$\frac{\partial^2 \Phi}{\partial v^2} = \frac{c_s^2}{v^3} > 0. \tag{10}$$

Thus, at any given r and T, Φ attains its minimum value when $v = c_s$:

$$\Phi_{\min} = c_s. \tag{11}$$

When solutions for Φ and T have been obtained as functions of r, Eq. (6) may be inverted to find $v(r)$:

$$v_{\pm} = \Phi \left\{ 1 \pm \left[1 - \left(\frac{c_s}{\Phi} \right)^2 \right]^{1/2} \right\}. \tag{12}$$

(Note that c_s is the *local* sound speed at the given level r.) Using the fact that $\Phi \geqslant \Phi_{\min}$, it is easy to show that $v_+ > c_s$ and $v_- < c_s$ (except at the critical point, where $v_+ = v_- = c_s$). Thus, for the transonic flows that interest us here, the desired solution branch is v_- for $r < r_c$ and v_+ for $r > r_c$. Figure 2 shows a sketch of Φ as a function of v, at a level r_1 where the sound speed is $c_s(r_1)$.

The requirement that $N = 0$ at $r = r_c$ can readily be shown to be equivalent to the condition that

$$\frac{d\Phi}{dr} = \frac{d\Phi_{\min}}{dr} \quad \text{at} \quad r = r_c. \tag{13}$$

For practical purposes, this condition yields a constraint on the free parameters of the problem, thus restricting the allowed wind solutions to a two-parameter family, characterized by, for example, the temperature and density at the base of the flow.

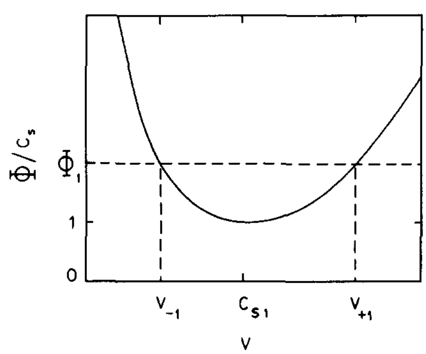

FIG. 2. A sketch of Φ as a function of v at a given radius r_1. Once the (unique) solution $\Phi_1 \equiv \Phi(r_1)/c_s(r_1)$ has been determined, Eq. (12) gives the two transonic branch values of the velocity: $v_{+1} \equiv v_+(r_1)$ and $v_{-1} \equiv v_-(r_1)$. These velocities become degenerate $[v_{+1} = v_{-1} = c_{s1} \equiv c_s(r_1)]$ at the critical point $(r_1 = r_c)$.

IV. Conclusion

We have seen that substitution of the new function Φ for the velocity as one of the dependent variables removes the critical point (and hence the numerical difficulties encountered there) from the set of coupled differential (wind) equations. This method has been used with great success in a study [6] of radiatively driven mass loss from the surface of X-ray bursting neutron stars. In such systems, it is crucial to know the outer boundary conditions as a function of the evolution of the deeper, hydrostatic layers of the neutron-star envelope. The simple and accurate technique described here for solving the equations of time-independent mass loss can be useful in other applications where such considerations are equally important.

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